Monotone Erasure Codes

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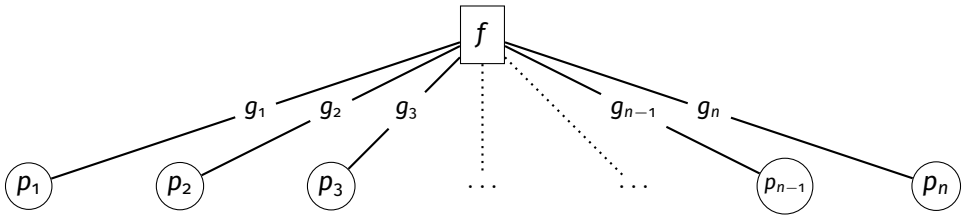
May 9, 2026

Problem

Goal: Store a file f among the nodes in a set $\mathcal{P} = \{p_1, p_2, \ldots, p_n\}$.

Each node p_i holds a piece of information g_i, called a **fragment** of the file f.



Certain groups of nodes in $\mathcal{P}$ should be able to reconstruct f.

Erasure Codes

An **erasure code** splits data into fragments so that the original data can still be recovered even if some fragments are lost.

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- *Maximum-distance separable (MDS)* codes (e.g., Reed-Solomon codes [RS60]) divide data into k pieces and encode them into $n > k$ fragments of equal size, where any k fragments allow reconstruction.

subsets of $\mathcal{P}$ of size at least k can reconstruct f

Threshold access structures

Access Structures

An **access structure** $\mathcal{A}$ on a set of nodes $\mathcal{P}$ is a collection of subsets of $\mathcal{P}$ such that no set is contained in another. Each set $A \in \mathcal{A}$ is called an **access set**.

Example: $\mathcal{A} = \{ \{a, b\}, \{b, c, d\}, \{b, c, e, f\} \}$

$\longrightarrow$ An access structure corresponds to a **monotone Boolean formula** (MBF).

Example: $(a \wedge b) \vee (b \wedge c \wedge d) \vee (b \wedge c \wedge e \wedge f)$

Access Structures

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- Secret sharing
- Voting power in blockchain systems
- Consensus protocols (e.g., XRP Ledger or Stellar network)

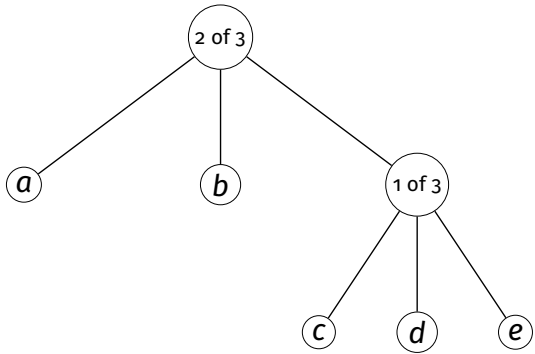
$\longrightarrow$ Access structures represent the underlying trust assumptions

$\longrightarrow$ Access sets specify which subset of nodes have enough power to perform certain actions

Representation of Access Structures

Let $\mathcal{A}$ be an access structure on $\mathcal{P}$

- An **access tree** of $\mathcal{A}$ is a rooted, labeled tree in which
  - internal vertex $\leftrightarrow$ threshold operator Θ_t^r (t of r)
  - leaf vertex $\leftrightarrow$ node in $\mathcal{P}$.



Contribution of this Work: Erasure Codes for any Access Structure

How can erasure codes be extended when the underlying assumption is captured by a general access structure?

$\longrightarrow$ Monotone erasure codes

$\longrightarrow$ Construction of *optimal* monotone erasure codes

$\longrightarrow$ Integrate monotone erasure codes into distributed protocols, e.g., AVID: Asynchronous Verifiable Information Dispersal [CT05]

Monotone Erasure Codes

Let $f \in \mathcal{F}$ be a file and let $\mathcal{G} = \mathcal{G}_1 \times \ldots \times \mathcal{G}_n$, where $\mathcal{G}_i$ are (possibly different) sets.

A **monotone erasure code** $\mathcal{C}$ for $\mathcal{A}$ on $\mathcal{P}$ is a scheme consisting of two functions:

$$f \in \mathcal{F} \xrightarrow{\text{Encode}} g = (g_1, g_2, \ldots, g_n) \in \mathcal{G}$$

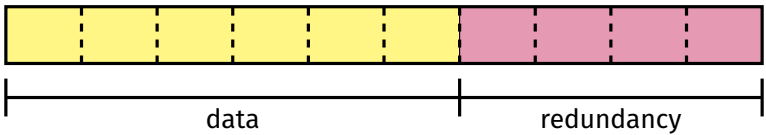
$g_1 \downarrow p_1$, $g_2 \downarrow p_2$, $g_n \downarrow p_n$ with $g_i \in \mathcal{G}_i$ the **fragment** of p_i

$$u \in \mathcal{G} \xrightarrow{\text{Decode}} \begin{cases} f' \in \mathcal{F} \\ \perp \end{cases}$$

Completeness of $\mathcal{C}$:
The nodes in each access set $A \in \mathcal{A}$ have enough information to reconstruct f, i.e., $f' = f$.

Overhead

The **overhead** β of $\mathcal{C}$ measures the amount of redundancy added to the original data during the encoding phase for tolerating loss of fragments.



$$\beta = \frac{\text{redundancy}}{\text{data}}$$

Efficiency: $\mathcal{C}$ should minimize the overhead

Linear Monotone Erasure Codes

An $[m, k]$-**linear monotone erasure code** $\mathcal{C}$ for $\mathcal{A}$ is a monotone erasure code for $\mathcal{A}$ that encodes files $f \in \mathbb{F}_q^k$ and produces fragments with combined size m with a linear operation.

ENCODE: $G \in \mathbb{F}_q^{k \times m}$ full-rank matrix with columns $(G_i)_{i \in [m]}$ and a labeling function $\phi : \{1, \ldots, m\} \to \mathcal{P}$

$$G = \left(\begin{array}{c|c|c|c} G_1 & G_2 & \ldots & G_m \\ \downarrow & \downarrow & & \downarrow \\ \phi(1) & \phi(2) & & \phi(m) \end{array} \right)$$

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$$G_{p_i} = \left(\begin{array}{c|c|c|c} G_{j_1} & G_{j_2} & \ldots & G_{j_{m_i}} \end{array} \right)$$

with $\phi(j_1) = \ldots = \phi(j_{m_i}) = p_i$

$$g_i = \begin{cases} f \cdot G_{p_i} & m_i > 0 \\ \perp & m_i = 0 \end{cases} \quad \text{(fragment of } p_i\text{)}$$

Linear Threshold Erasure Codes

Let $\mathcal{A}$ be a threshold access structure on $\mathcal{P}$:

$$\mathcal{A} = \{A \in \mathcal{A} \mid |A| = w\} \text{ for some } w \in \mathbb{N}$$

An $[m, k]$-**linear threshold erasure code** $\mathcal{C}$ for $\mathcal{A}$ is an $[m, k]$-linear monotone erasure code where

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$$G = \begin{pmatrix} 1 & 1 & \dots & 1 \\ \alpha_1 & \alpha_2 & \dots & \alpha_m \\ \vdots & & & \vdots \\ \alpha_1^{k-1} & \alpha_2^{k-1} & \dots & \alpha_m^{k-1} \end{pmatrix} \in \mathbb{F}_q^{k \times m}.$$

$$\phi(1) \downarrow \quad \phi(2) \downarrow \quad \phi(n) \downarrow$$

$$p_1 \quad p_2 \quad p_n$$

where G is a Vandermonde matrix with pairwise distinct $\alpha_1, \dots, \alpha_m \in \mathbb{F}_q^m$.

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Linear Threshold Erasure Codes $\leftrightarrow$ MDS codes

Optimal Linear Monotone Erasure Codes

Given an access structure $\mathcal{A}$ on a set of nodes $\mathcal{P}$, the goal is to find

- Parameters m, k
  - m: Combined size of all fragments
  - k: File size
- Labeling function ϕ
- Encoding matrix G

defining an $[m, k]$-linear monotone erasure code $\mathcal{C}$ that is complete and has *minimal* storage overhead

$$\beta = \frac{m - k}{k}$$

Finding the Optimal Parameters

For a given access strucure $\mathcal{A}$

Goal: Find parameters m, k of a linear monotone erasure code $\mathcal{C}$ such that

$$\frac{m}{k} = \frac{1}{k} \sum_{i=1}^{n} m_i$$

is minimized and it holds

$$\sum_{p_i \in A} m_i \geq k \quad \forall A \in \mathcal{A}$$

$$y_i = \frac{m_i}{k} \longrightarrow$$

Linear Programming Problem (LPP):

Find $y = (y_1, \ldots, y_n) \in \mathbb{Q}^n_{\geq 0}$ such that

$$\min \sum_{i=1}^{n} y_i$$

$$\text{subject to} \sum_{p_i \in A} y_i \geq 1 \quad \forall A \in \mathcal{A}$$

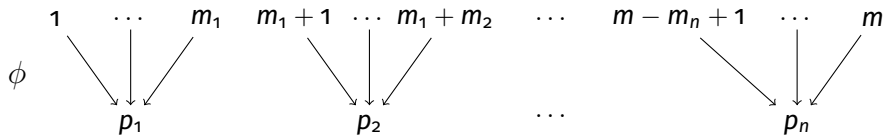
Scale to integer solution: choose $k =$ lcm of denominators of all y_i.

Building the Code from a Threshold Code

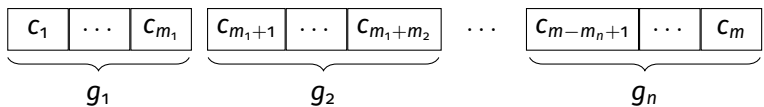
With optimal m, k and m_i for $i \in [n]$, we build an $[m, k]$-linear monotone erasure code:

Base code. $[m, k]$-linear threshold erasure code $\mathcal{C}'$ (MDS-code) over $\mathbb{F}_q$, where $q \geq m$

Labeling Function.



Encoding.

$\text{MDSEncode}(f) \to f \cdot G =$ 

Decoding. Corresponds to MDSDecode

Building the Code from a Threshold Code

Solving LPP is efficient if $\mathcal{A}$ has a compact description in the number of nodes

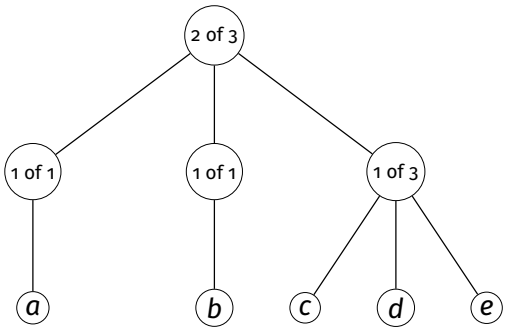
$\longrightarrow$ But there may be exponentially many (in n) access sets in $\mathcal{A}$!

Goal: Find the optimal parameters of the base code without solving the corresponding LP problem for a given access structure $\mathcal{A}$.

Partitioned Threshold Access Structures

A **partitioned threshold access structure** $\mathcal{A}$ on $\mathcal{P}$ is given by an access tree T where every node in $\mathcal{P}$ appears at most once in T.

- Consensus protocol in the Stellar blockchain (*https://stellar.org*): Each organization has its own nodes, i.e., the node set $\mathcal{P}$ is partitioned into organizations $B_1, \ldots, B_\delta$.



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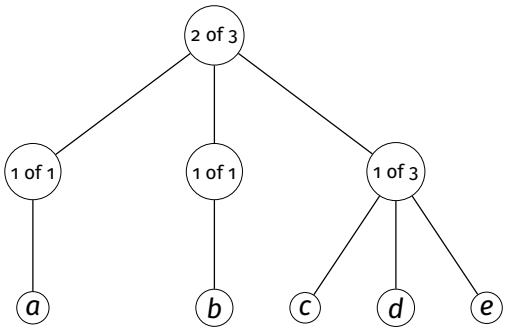
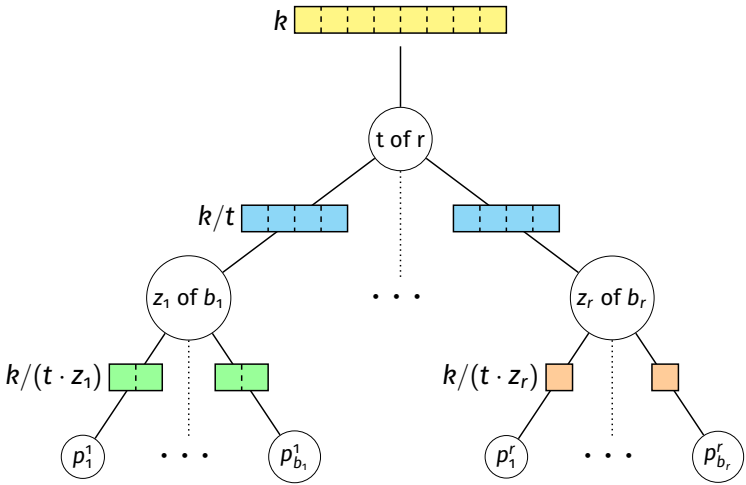


A partitioned access structure is called **L-level partitioned threshold access structure** if all leaves have depth L in the tree.

Uniform Fragment Assignment

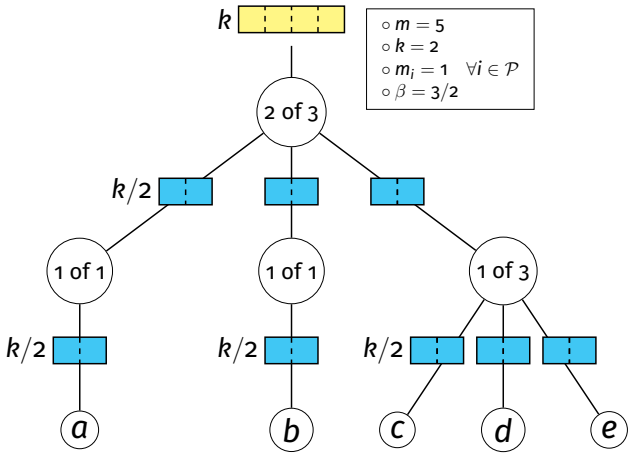


Given a 2-level partitioned access structure:

Base code $\mathcal{C}'$ has parameters:

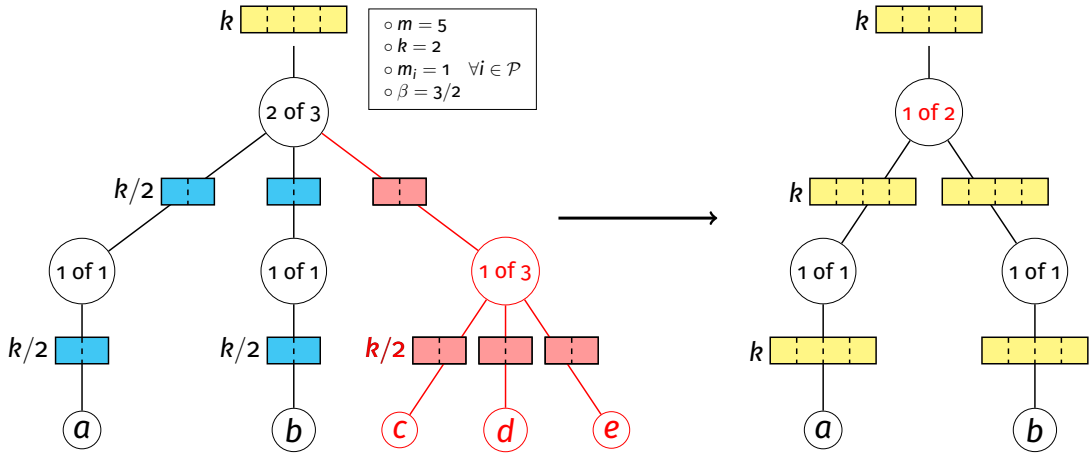
- $m = \sum_{i=1}^{r} b_i \cdot k/(t \cdot z_i)$
- $k = lcm(t \cdot z_1, \ldots, t \cdot z_r)$

Example - Tree Reduction



- $m = 5$
- $k = 2$
- $m_i = 1 \quad \forall i \in \mathcal{P}$
- $\beta = 3/2$

Example - Tree Reduction



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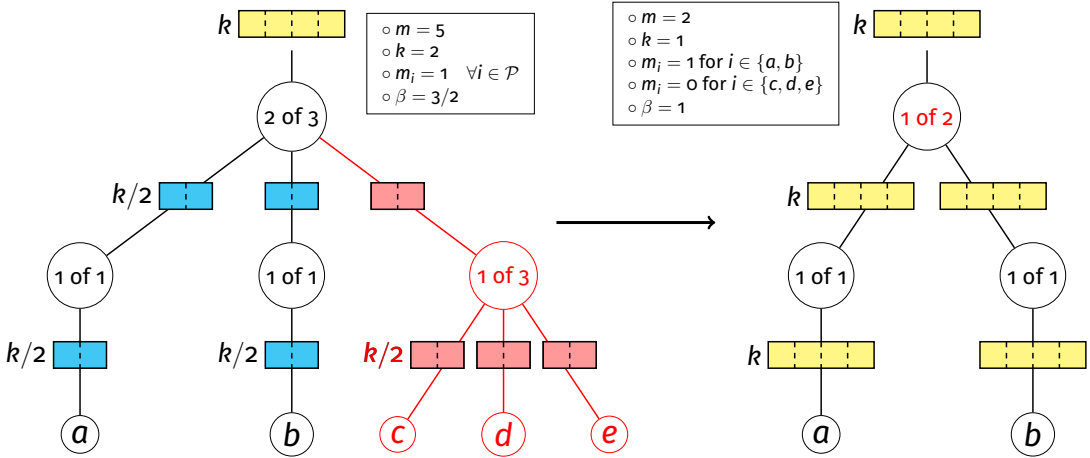


Algorithm

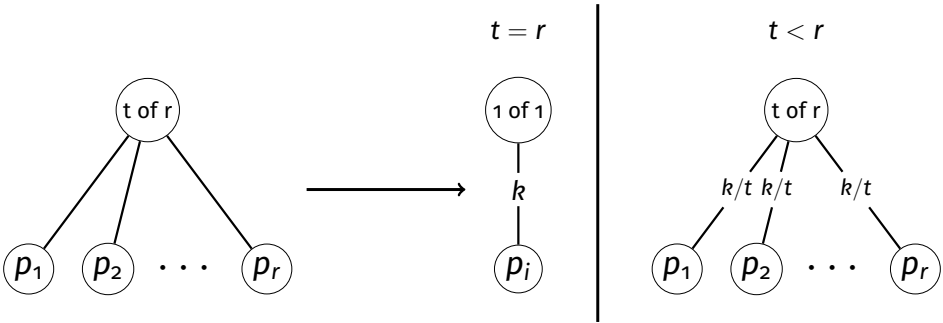
Let T be an access tree of an L-level partitioned threshold access structure

Observation: Overhead is minimized if all subtrees of T have an optimal fragment assignment

$\rightarrow$ Reduce T by pruning all subtrees with negative overhead contribution recursively in a depth-wise manner

$\rightarrow$ Applying uniform assignment to the reduced tree yields the optimal parameters of the base code

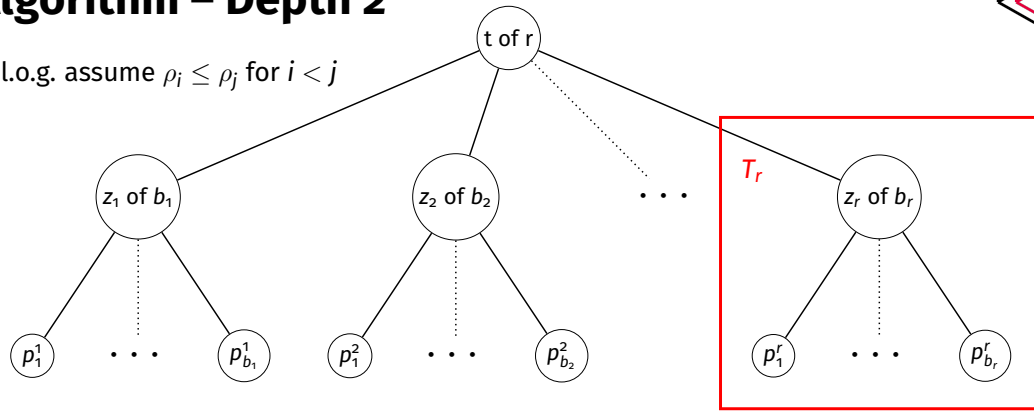
Algorithm – Depth 1



for each T_i compute $\rho_i = \frac{r}{t}$ (cost of T_i)

Algorithm – Depth 2

W.l.o.g. assume $\rho_i \leq \rho_j$ for $i < j$



Remove T_r if $t > 1$ and $\rho_r \geq \underbrace{\left(\frac{t}{t-1} - 1\right) \sum_{i=1}^{r-1} \rho_i}_{\text{additional cost}}$

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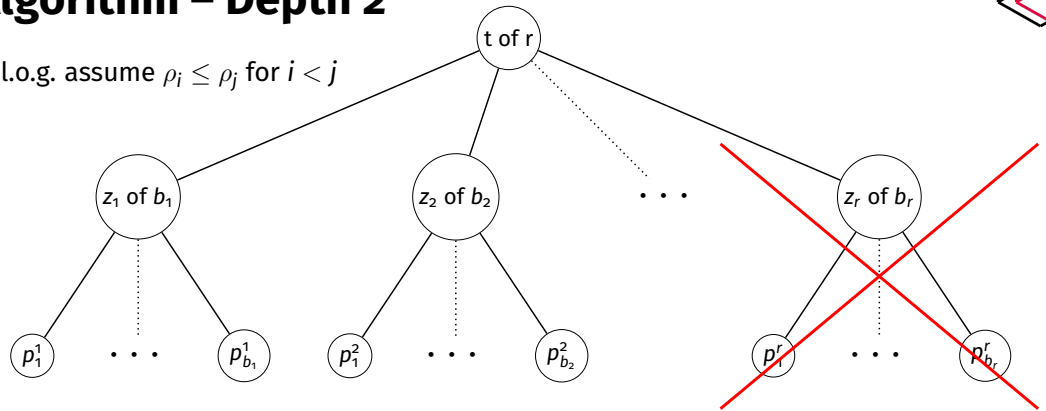

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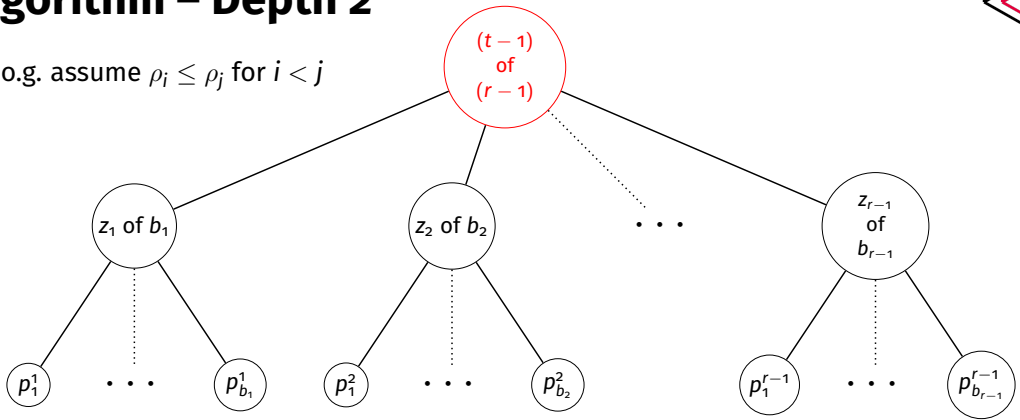
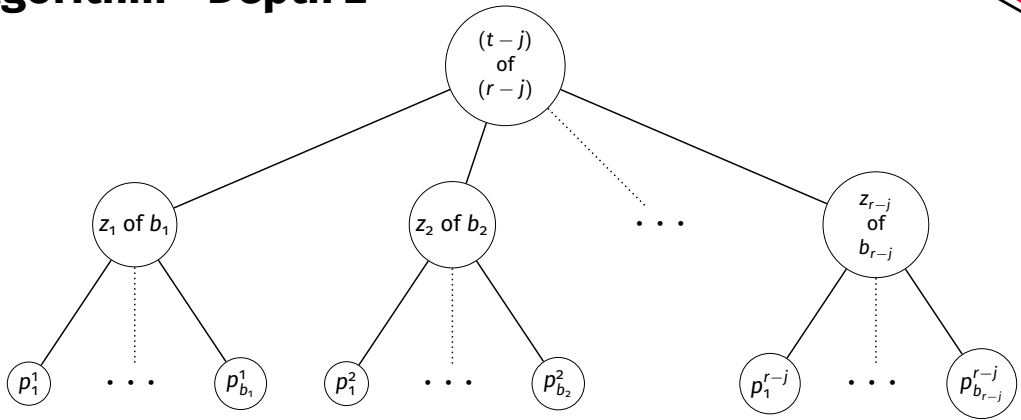


Remove T_{r-1} if $t - 1 > 1$ and $\rho_{r-1} \geq \underbrace{\left(\frac{t-1}{t-2} - 1\right) \sum_{i=1}^{r-2} \rho_i}_{\text{additional cost}}$

Algorithm - Depth 2



Cost of T': $\rho_{T'} = \frac{1}{t-j} \sum_{i=1}^{r-j} \rho_i$ where $j \in \mathbb{N}$ is the number of pruned subtrees

Algorithm for any Depth

- Apply the procedure to all depth-3 subtrees
- Replace each subtree with a reduced tree and compute its cost
- Repeat recursively until T is fully reduced (if possible)

Applications

Goal: Allow a node to distribute data among n nodes such that it can always be recovered correctly despite Byzantine failures.

AVID [CT05]

- Up to t Byzantine nodes
- Threshold Byzantine quorum systems

Classical erasure code

GAVID

- Subsets of nodes, specified by the *fail-prone system*, may be Byzantine
- General Byzantine quorum systems [HM00; MR97]

Monotone erasure code

Future Work

- *Efficient* construction of optimal monotone erasure codes for *arbitrary* monotone access structures
- Constructing “flexible” monotone erasure codes that adapt to changing access structures
- Investigate how other methods from coding theory (e.g., rateless codes or low-density parity-check codes) can be employed in the setting of monotone erasure codes

Thank you for your attention

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May 9, 2026

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Weighted Access Structure

Proof-of-stake blockchains use a **weighted** access structure $\mathcal{A}$: each node $p_i \in \mathcal{P}$ has a weight w_i.

Access sets: $A \subseteq \mathcal{P}$ such that $\sum_{p_i \in A} w_i > W$, for some $W \in \mathbb{R}$.

Then:

- $\mathcal{A}$ is monotone, so we can apply our algorithm to find an optimal monotone erasure code for $\mathcal{A}$
- $\mathcal{A}$ is not partitioned, so the algorithm is inefficient

Open problem: Find an efficient way to build an optimal linear monotone erasure code for $\mathcal{A}$